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$9j$-symbols of the oscillator algebra and Krawtchouk polynomials in two variables

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Abstract. A simple generating function for the $9j$-symbols of the oscillator algebra is found. On the basis of this function it is shown that corresponding $9j$-symbols are expressed in terms of polynomials $Q_{pr}(k,m)$ in two discrete variables which are orthogonal with respect to the trinomial distribution. These polynomials depend on four independent parameters and can be considered as a two-dimensional analogue of the Krawtchouk polynomials. Difference–difference relations, the factorization chain, the duality property, and the Rodriguez formula for these polynomials are obtained. It is shown that the polynomials $Q_{pr}(k,m)$ are eigenfunctions of two commuting difference operators. These polynomials are also covariant with respect to two commuting difference derivation operators. In a special symmetric case these polynomials admit a simple factorized expression in terms of two distinct ordinary Krawtchouk polynomials. In another special case we obtain an explicit expression of $9j$-symbols in terms of the Appell hypergeometric function F_1 in two variables.

1. Introduction

$3nj$-symbols of Lie algebras are very useful tools in numerous theoretical problems. The importance of the study of such objects for pure mathematics and mathematical physics can be illustrated by finding the Askey–Wilson polynomials [1] which are believed to be the 'most general' orthogonal polynomials having nice properties. These polynomials were discovered, in particular, on the basis of known properties of $6j$-symbols for the $su(2)$ algebra (concerning the connection between these objects see, e.g., [2–4]). Note that more simple $3j$-symbols (or Clebsch–Gordan coefficients) of the $su(2)$ algebra are expressed in terms of the Hahn polynomials [4].

Suslov showed [5, 6] that more complicated $9j$-symbols of the $su(2)$ algebra can be expressed in terms of some orthogonal polynomials in *two* discrete variables. However, an explicit expression for such polynomials is yet unknown.

On the other hand, some classes of orthogonal multivariable polynomials (and generally non-polynomial multivariable functions) are now being intensively studied with respect to quantum integrable systems and combinatorical problems (see, e.g., [7–10]). See also [11] for the connection of multivariate orthogonal polynomials with representations of Lie algebras.

In this paper we show that $9j$-symbols of the oscillator algebra (which can be considered as a contraction of the $su(2)$ algebra) can be expressed in terms of Krawtchouk polynomials in two discrete arguments. The main tool in our analysis will be a (perhaps new) generating function for $9j$-symbols. This generating function is obtained on the basis of our approach to constructing $3nj$-symbols [12]. In the particular case of the symmetric condition for the

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representation parameters we arrive at the simple Krawtchouk polynomials in two variables which have been studied by Prizva [13].

It is well known that both $3j$- and $6j$-symbols of the oscillator algebra are expressed in terms of the ordinary Krawtchouk polynomials in one discrete variable [14, 15]. So our result can be considered as a non-trivial extension of this result to the polynomials in two variables.

The paper is organized as follows. In section 2 we recall necessary facts concerning the oscillator algebra and its $9j$-symbols. In section 3 we apply our method [12] to the construction of a generating function for $9j$-symbols. In section 4 we derive two pairs of difference–difference relations and show that the oscillator $9j$-symbols can be expressed in terms of orthogonal polynomials $Q_{pr}(k, m; N)$ in two discrete variables, which are orthogonal with respect to the trinomial distribution. In section 5 a factorization technique for the polynomials $Q_{pr}(k, m; N)$ is proposed. This technique is also known in the mathematical literature as the Darboux transformation method. On the basis of this technique, we find two families of ladder operators allowing transformation of a polynomial $Q_{pr}(k, m; N)$ to any polynomial $Q_{p',r'}(k, m; N')$ with other arbitrary parameters p', r' and N'. In particular, we construct a simple Rodriguez-type formula for these polynomials. We also show that the polynomials $Q_{pr}(k, m; N)$ are eigenfunctions of two commuting difference operators. In section 6 we propose another algebraic scheme based on the observation that the operators introduced in section 4 form a linear (Lie) algebra under commutations, whereas the corresponding Hamiltonians form nonlinear (quadratic) algebra. This algebra plays a role of a hidden symmetry algebra of the eigenvalue problem for the $9j$-symbols (and corresponding polynomials $Q_{pr}(k, m; N)$). In section 7 we derive the duality property of the polynomials $Q_{pr}(k, m; N)$. In section 8 we consider a special choice of the algebras' parameters and find in this case an explicit expression for the polynomials $Q_{pr}(k, m; N)$ in terms of the product of two ordinary Krawtchouk polynomials. In section 9 it is shown that in another special case $p + r = N$, the $9j$-symbols admit an explicit expression in terms of the Appell hypergeometric function F_1 in two variables.

2. Oscillator algebra, its addition rule and $9j$-symbols

The oscillator algebra is described by the commutation relations (we adopt a notation which is slightly different from the standard one [16])

$$[A_-, A_+] = a \qquad [A_0, A_\pm] = \pm A_\pm \tag{2.1}$$

where $A_0, A_\pm$ are three generators, a is a positive constant. Note that if one considers a in (2.1) as not a constant but rather as a fourth generator (commuting with the others) we then obtain the standard Heisenberg Lie algebra with four generators $A_0, A_\pm, a$ (see, e.g., [17]). We prefer, however, to use the term 'oscillator algebra' in order to stress that a is considered as an independent constant. (This can be achieved if one is restricted to the representations of the Heisenberg algebra with a fixed value of the generator a.) The Casimir operator of the oscillator algebra has the expression

$$Q = aA_0 - A_+A_-. \tag{2.2}$$

Unitary irreducible representations of the algebra (2.1) are described by the basis $|n; a; \rho\rangle$ for which the action of the generators is described by

$$\begin{aligned} A_0|n; a; \rho\rangle &= (n + \rho)|n; a; \rho\rangle \\ A_-|n; a; \rho\rangle &= \sqrt{an}|n - 1; a; \rho\rangle \end{aligned} \tag{2.3}$$

$$A_+|n; a; \rho\rangle = \sqrt{a(n+1)}|n+1; a; \rho\rangle$$
$$n = 0, 1, \ldots \tag{2.4}$$

where the representation parameter ρ is defined by the value of the Casimir operator on the given representation: $Q|n; a; \rho\rangle = \rho a|n; a; \rho\rangle$. The operators A_- and A_+ are Hermitian conjugated whereas the operator A_0 is Hermitian given the representation (2.3). Note that the ordinary Bose operators are a special case of the representations of the algebra (2.1) with $a = 1$, $\rho = 0$.

Now consider the addition of two independent oscillator algebras $A_0^{(i)}, A_\pm^{(i)}, a_i$ and $A_0^{(k)}, A_\pm^{(k)}, a_k$ [16], where a_i, a_k are two different arbitrary positive parameters, and all the operators with superscript i commute with the operators with superscript k. The addition rule has the form

$$A_p^{(ik)} = A_p^{(i)} + A_p^{(k)} \qquad a_{ik} = a_i + a_k \tag{2.5}$$

where $p = 0, \pm$ and (super)subscript ik indicates the new oscillator algebra which is obtained by adding of the two algebras.

The Casimir operator of the resulting algebra has the expression

$$Q_{ik} = a_i A_0^{(k)} + a_k A_0^{(i)} - A_+^{(i)} A_-^{(k)} - A_+^{(k)} A_-^{(i)} + Q_i + Q_k \tag{2.6}$$

where Q_i and Q_k are the Casimir operators of the adding algebras.

One can introduce the coupled basis $|n_{ik}; a_{ik}, \rho_{ik}\rangle$ for the resulting algebra where the notation is the same as in (2.3). From the adding rules (2.5) we have the obvious relation

$$n_{ik} + \rho_{ik} = n_i + n_k + \rho_i + \rho_k \tag{2.7}$$

where the coupled representation parameter ρ_{ik} takes the values

$$\rho_{ik} = \rho_i + \rho_k + p \qquad p = 0, 1, 2 \ldots. \tag{2.8}$$

From (2.8) it is clear that the representation of the resulting algebra in coupled basis is uniquely defined by fixing of the six parameters $\rho_i, \rho_k, a_i, a_k, n_{ik}$ and p.

The Clebsch–Gordan decomposition has the form

$$|n_{ik}; a_{ik}; \rho_{ik}\rangle = \sum_{n_i, n_k} C(n_{ik}, p; n_i, \rho_i, n_k, \rho_k)|n_i; a_i; \rho_i\rangle|n_k; a_k \rho_k\rangle \tag{2.9}$$

where the sum on the right-hand side of (2.9) is restricted by the conditions $n_i + n_k = n_{ik} + p$ and $0 \leqslant n_i \leqslant n_{ik} + p$. The Clebsch–Gordan coefficients (CGC) $C(n_{ik}, p; n_i, \rho_i, n_k, \rho_k)$ can be easily expressed in terms of Krawtchouk polynomials (see, e.g., [14, 15]). In what follows we need only an explicit expression for the 'vacuum' CGC

$$C_n(0) = C(0, p; n, \rho_i, p - n_i, \rho_k) = (-1)^n \sqrt{\frac{a_i^p\, p!}{a_{ik}^p n!(p-n)!}}(a_k/a_i)^{n/2}. \tag{2.10}$$

In an analogous manner, one can consider adding three and more oscillator algebras leading to $3nj$-symbols. For example, for three algebras we have two possible addition schemes $((1 \oplus 2) \oplus 3)$ and $(1 \oplus (2 \oplus 3))$. For these two schemes we have the corresponding two coupled bases $|n_{12}, a_{12}, \rho_{12}; \rho_3, \rho_{123}\rangle$ and $|n_{23}, a_{23}, \rho_1, \rho_{23}; \rho_{123}\rangle$, where ρ_{123} corresponds to the representation parameter of the resulting algebra. Decomposition

$$|*, a_{23}, \rho_{23}; \rho_1, \rho_{123}\rangle = \sum_{\rho_{12}, \rho_{23}} R(\rho_{12}, \rho_{23}; \rho_1, \rho_2, \rho_3, \rho_{123})|*, a_{12}, \rho_{12}; \rho_3, \rho_{123}\rangle \tag{2.11}$$

leads to the Racah coefficients (or 6j-symbols) $R(\rho_{12}, \rho_{23}; \rho_1, \rho_2, \rho_3, \rho_{123})$. (We denote by $*$ the first argument in the bases in (2.11) because the Racah decomposition is invariant under a change of the value n_{ik}.) Note also that the sum on the right-hand side of (2.11) is really reduced to one summation because of the restrictions

$$\rho_{12} = \rho_1 + \rho_2 + p_{12}, \rho_{23} = \rho_2 + \rho_3 + p_{23}$$

$$\rho_{123} = \rho_3 + \rho_{12} + p_{12,3} = \rho_1 + \rho_{23} + p_{1,23} \tag{2.12}$$

$$p_{12} + p_{12,3} = p_{23} + p_{1,23} = N = \rho_{123} - \rho_1 - \rho_2 - \rho_3. \tag{2.13}$$

Hence, we can rewrite the Racah decomposition in the following concise form,

$$|*; p_{23}, p_{1,23}\rangle = \sum_{p_{12}, p_{12,3}} R_{p_{12} p_{12,3}}(p_{23}, p_{1,23})|*; p_{12}, p_{12,3}\rangle \tag{2.14}$$

where the restrictions (2.13) are assumed and we omit for brevity the explicit dependence of the Racah coefficients on the parameters ρ_1, ρ_2, ρ_3 and ρ_4. The Racah coefficients for the oscillator algebra can also be expressed in terms of the Krawtchouk polynomials [14, 15].

Consider adding four different oscillator algebras. There are two different schemes: $((1 \oplus 2) \oplus (3 \oplus 4))$ and $((1 \oplus 3) \oplus (2 \oplus 4))$. This leads to the decomposition

$$|*; p_{13}, p_{24}, p_{13,24}\rangle = \sum_{p_{12}, p_{34}, p_{12,34}} F^{p_{13}, p_{24}, p_{13,24}}_{p_{12}, p_{34}, p_{12,34}} |*; p_{12}, p_{34}, p_{12,34}\rangle \tag{2.15}$$

where the sum on the right-hand side of (2.15) is really reduced to a two-fold sum, because of the restrictions

$$p_{12} + p_{34} + p_{12,34} = p_{13} + p_{24} + p_{13,24} = N = \rho_{1234} - \rho_1 - \rho_2 - \rho_3 - \rho_4. \tag{2.16}$$

In what follows we redenote indices for brevity: $p_{13} = p$, $p_{24} = q$, $p_{13,24} = r$, $p_{12} = k$, $p_{34} = l$, $p_{12,34} = m$. Then the decomposition (2.15) is rewritten as

$$|*; p, q, r\rangle = \sum_{klm} F^{pqr}_{klm} |*; k, l, m\rangle \tag{2.17}$$

with the restrictions

$$p + q + r = k + l + m = N. \tag{2.18}$$

The Fano coefficients (9j-symbols) F^{pqr}_{klm} can also depend on the representation parameters ρ_i, $i = 1, 2, 3, 4$. We will see, however, that really these coefficients depend only on the algebras parameters a_i. The dependence on ρ_{1234} is replaced by a dependence on the value of the integer parameter N, as is seen from (2.16).

3. Generating function

In this section we apply our method [12] for calculating the generating function of the 9j-symbols.

Consider the coherent states $|z; a; \rho\rangle$ for the oscillator algebra defined as eigenstates of the annihilation operator

$$A_-|z\rangle = z|z\rangle. \tag{3.1}$$

Hence, we have expansion coefficients in terms of the standard basis

$$\langle n|z\rangle = \frac{z^n}{\sqrt{n!a^n}} e^{-|z|^2/2a}. \tag{3.2}$$

In what follows we denote

$$|z_1, z_2\rangle = |z_1; a_1; \rho_1\rangle \otimes |z_2; a_2; \rho_2\rangle$$

$$|z_1, z_2, z_3\rangle = |z_1; a_1; \rho_1\rangle \otimes |z_2; a_2; \rho_2\rangle \otimes |z_3; a_3; \rho_3\rangle \qquad \text{etc.}$$

Using (3.2) and the 'vacuum' CGC (2.10) we easily find the following matrix element,

$$\langle z_1, z_2|0; a_{12}, \rho_{12}\rangle = \mu_{12}(p)\left(\frac{z_2}{a_2} - \frac{z_1}{a_1}\right)^p \langle z_1|0_1\rangle\langle z_2|0_2\rangle \tag{3.3}$$

where $\langle z_i|0_i\rangle = \mathrm{e}^{-|z_i|^2/2a_i}$ and

$$\mu_{ik}(p) = \left(\frac{a_i a_k}{a_{ik}}\right)^{p/2} \frac{1}{\sqrt{p!}}. \tag{3.4}$$

Quite similarly, for two schemes of the adding of four algebras we can write down the following matrix elements,

$$\begin{aligned}\langle z_1, z_2, z_3, z_4|0; p, q, r\rangle &= \mu_{12}(p)\mu_{34}(q)\mu_{12,34}(r)\\ &\quad\times \prod_{i=1}^{4}\langle z_i|0_i\rangle \left(\frac{z_2}{a_2} - \frac{z_1}{a_1}\right)^p \left(\frac{z_4}{a_4} - \frac{z_3}{a_3}\right)^q \left(\frac{z_3+z_4}{a_{34}} - \frac{z_1+z_2}{a_{12}}\right)^r \end{aligned}\tag{3.5}$$

$$\begin{aligned}\langle z_1, z_2, z_3, z_4|0; k, l, m\rangle &= \mu_{13}(k)\mu_{24}(l)\mu_{13,24}(m)\\ &\quad\times \prod_{i=1}^{4}\langle z_i|0_i\rangle \left(\frac{z_3}{a_3} - \frac{z_1}{a_1}\right)^k \left(\frac{z_4}{a_4} - \frac{z_2}{a_2}\right)^l \left(\frac{z_2+z_4}{a_{24}} - \frac{z_1+z_3}{a_{13}}\right)^m \end{aligned}\tag{3.6}$$

where we adopt the same notation as in (2.17).

From (2.17), (3.5) and (3.6) we immediately obtain the identity

$$\begin{aligned}&\left(\frac{z_2}{a_2} - \frac{z_1}{a_1}\right)^p \left(\frac{z_4}{a_4} - \frac{z_3}{a_3}\right)^q \left(\frac{z_3+z_4}{a_{34}} - \frac{z_1+z_2}{a_{12}}\right)^r\\ &\qquad = \sum_{klm} \tilde{F}^{pqr}_{klm} \left(\frac{z_3}{a_3} - \frac{z_1}{a_1}\right)^k \left(\frac{z_4}{a_4} - \frac{z_2}{a_2}\right)^l \left(\frac{z_2+z_4}{a_{24}} - \frac{z_1+z_3}{a_{13}}\right)^m\end{aligned} \tag{3.7}$$

where $\tilde{F}^{pqr}_{klm}$ are defined as

$$\tilde{F}^{pqr}_{klm} = \frac{\mu_{13}(k)\mu_{24}(l)\mu_{13,24}(m)}{\mu_{12}(p)\mu_{34}(q)\mu_{12,34}(r)} F^{pqr}_{klm}. \tag{3.8}$$

The identity (3.7) should be valid for all values of the coherent parameters z_i. Using this freedom we can choose the following parametrization,

$$\frac{z_3}{a_3} - \frac{z_1}{a_1} = \frac{a_2 a_4 a_{13}}{a_1 a_3 a_{24}} u \qquad \frac{z_4}{a_4} - \frac{z_2}{a_2} = 1 \qquad \frac{z_2+z_4}{a_{24}} - \frac{z_1+z_3}{a_{13}} = \frac{a_4}{a_{24}} v \tag{3.9}$$

where u and v are two independent variables. Then the identity (3.7) becomes

$$\Psi^{pr}(u, v; N) = (1 - v - \alpha u)^p (1 + \beta v - \gamma u)^q (1 + u + \delta v)^r = \sum_{klm} \Phi^{pqr}_{klm} u^k v^m \tag{3.10}$$

where $\alpha = a_2/a_1$, $\beta = a_4/a_2$, $\gamma = a_4/a_3$, $\delta = (a_1 a_4 - a_2 a_3)/a_2 a_5$, $a_5 = a_1 + a_2 + a_3 + a_4$ and

$$\Phi^{pqr}_{klm} = \tilde{F}^{pqr}_{klm} \left(\frac{a_2 a_4 a_{13}}{a_1 a_3 a_{24}}\right)^k \left(\frac{a_4}{a_{24}}\right)^m \left(-\frac{a_{24}}{a_4}\right)^p \left(\frac{a_{24}}{a_2}\right)^q \left(\frac{a_{12} a_{34} a_{24}}{a_2 a_4 a_5}\right)^r \tag{3.11}$$

are modified $9j$-symbols. Note that the parameters α, β, γ and δ are not independent because of the relation $\delta = \beta(\gamma - \alpha)/(\gamma + \gamma\alpha + \alpha\beta + \gamma\alpha\beta)$.

Formula (3.10) yields the generating function $\Psi^{pr}(u, v; N)$ for the $9j$-symbols of the oscillator algebra. This formula can be exploited in order to obtain many useful relations for $9j$-symbols.

4. 9j-symbols and orthogonal polynomials in two discrete variables

We now show that $9j$-symbols can be expressed in terms of some orthogonal polynomials in two discrete variables. Indeed, let us represent modified $9j$-symbols in the form

$$\Phi_{klm}^{pqr} = Q_{pr}(k, m; N)\Phi_{klm}^{0N0} \tag{4.1}$$

where $Q_{pr}(k, m; N)$ are unknown functions. The 'vacuum amplitude' Φ_{klm}^{0N0} is easily found from (3.10) when $p = r = 0$, $q = N$:

$$\Phi_{klm}^{0N0} = \frac{N!}{k!l!m!}(-\gamma)^k(\beta)^m. \tag{4.2}$$

Let us show that $Q_{pr}(k, m; N)$ are orthogonal polynomials in two discrete variables k and m. For this goal we derive four difference–difference (recurrence) relations for the functions $Q_{pr}(k, m; N)$.

The first recurrence relation follows from an obvious formula

$$(1 - v - \alpha u)\Psi^{pr}(u, v; N) = \Psi^{p+1,r}(u, v; N + 1). \tag{4.3}$$

Expanding the left- and right-hand sides of (4.3) in terms of u and v and using (3.10) and (4.1) we obtain the relation

$$L^+(N)Q_{pr}(k, m; N - 1) = NQ_{p+1,r}(k, m; N) \tag{4.4}$$

where $L^+(N)$ is the difference operator acting on the space of functions in two variables k and m by the formula

$$L^+(N) = \frac{\alpha k}{\gamma}T_k^- - \frac{m}{\beta}T_m^- + N - k - m \tag{4.5}$$

where we have introduced elementary shift operators defined as $T_k^{\pm}Q(k, m) = Q(k \pm 1, m)$, $T_m^{\pm}Q(k, m) = Q(k, m \pm 1)$.

Analogously from

$$(1 + u + \delta v)\Psi^{pr}(u, v; N) = \Psi^{p,r+1}(u, v; N + 1) \tag{4.6}$$

we get another difference relation

$$M^+(N)Q_{pr}(k, m; N - 1) = NQ_{p,r+1}(k, m; N) \tag{4.7}$$

where

$$M^+(N) = -\frac{k}{\gamma}T_k^- + \frac{\delta m}{\beta}T_m^- + N - k - m. \tag{4.8}$$

Relations (4.4) and (4.7) are the first pair of the relations allowing to raise the parameter N.

In order to get the second pair of such relations lowering the parameter N we start with the easily verified differential relations for the generating function (in what follows by $\Psi_u(u, v)$ and $\Psi_v(u, v)$ we mean derivatives with respect to the corresponding variables)

$$\begin{aligned}(\delta - \beta - u(\beta + \gamma\delta))\Psi_u^{pr}(u, v; N) - (1 + \gamma + v(\beta + \gamma\delta))\Psi_v^{pr}(u, v; N)\\ +N(\beta + \gamma\delta)\Psi^{pr}(u, v; N) = \frac{p\varepsilon}{1 - v - \alpha u}\Psi^{pr}(u, v; N)\\ = p\varepsilon\Psi^{p-1,r}(u, v; N - 1)\end{aligned} \tag{4.9}$$

and

$$(1+\beta-u(\alpha\beta+\gamma))\Psi_u^{pr}(u,v;N)+(\gamma-\alpha-v(\alpha\beta+\gamma))\Psi_v^{pr}(u,v;N)$$
$$+N(\alpha\beta+\gamma)\Psi^{pr}(u,v;N)=\frac{r\varepsilon}{1+u+\delta v}\Psi^{pr}(u,v;N)$$
$$=r\varepsilon\Psi^{p,r-1}(u,v;N-1) \qquad (4.10)$$

where

$$\varepsilon=1+\beta+\gamma+\delta(\gamma-\alpha)+\alpha\beta. \qquad (4.11)$$

From (4.9) and (4.10) we find the following recurrence relations

$$L^-Q_{pr}(k,m;N)=\frac{p\varepsilon}{N}Q_{p-1,r}(k,m;N-1) \qquad (4.12)$$

and

$$M^-Q_{pr}(k,m;N)=\frac{r\varepsilon}{N}Q_{p,r-1}(k,m;N-1) \qquad (4.13)$$

where the operators L^- and M^- are defined by

$$L^-=\gamma(\beta-\delta)T_k^+-\beta(1+\gamma)T_m^++\beta+\gamma\delta \qquad (4.14)$$
$$M^-=-\gamma(1+\beta)T_k^++\beta(\gamma-\alpha)T_m^++\alpha\beta+\gamma. \qquad (4.15)$$

From the relations (4.4), (4.7), (4.12) and (4.13) we can construct successively all the functions $Q_{pr}(k,m;N)$ starting from the trivial vacuum function $Q_{00}(k,m;N)=1$. It is easily seen that then we indeed arrive at polynomials in two discrete variables k and m having degree $p+r$ (as usual, by the degree of the polynomial in two variables x and y we mean the maximal value $s+t$ among all the monomials x^sy^t in the expansion of the polynomial).

In order to find the orthogonality property of the polynomials $Q_{pr}(k,m;N)$ we return to the 'true' $9j$-symbols having the analogous representation

$$F_{klm}^{pqr}=F_{klm}^{0N0}P_{pr}(k,m;N) \qquad (4.16)$$

where again $P_{pr}(k,m;N)$ are polynomials in two discrete arguments k and m and the 'vacuum amplitude' can be calculated from (4.2), (3.11) and (3.8):

$$F_{klm}^{0N0}=(-1)^k\sqrt{\frac{N!}{k!l!m!}}\left(\frac{a_3a_4}{a_{13}a_{34}a_{24}}\right)^{N/2}(a_1a_{24}/a_3)^{k/2}(a_2a_{13}/a_4)^{l/2}a_5^{m/2}. \qquad (4.17)$$

The orthogonality property for the polynomials $P_{pr}(k,m;N)$ follows from the corresponding orthogonality properties of the $9j$-coefficients,

$$\langle p'q'r'|pqr\rangle=\sum_{klm}F_{klm}^{pqr}F_{klm}^{p'q'r'}=\delta_{pp'}\delta_{rr'}. \qquad (4.18)$$

Indeed, substituting (4.16) into (4.18) we get the orthogonality relation

$$\sum_{km}w_{km}P_{pr}(k,m;N)P_{p'r'}(k,m;N)=\delta_{pp'}\delta_{rr'} \qquad (4.19)$$

where the weight function is

$$w_{km}=(F_{klm}^{0N0})^2. \qquad (4.20)$$

It is easily seen from (4.17) that this weight function coincides with trinomial distribution

$$w_{km}=\left(\frac{N!}{k!l!m!}\right)b_1^kb_2^lb_3^m \qquad l=N-k-m \qquad (4.21)$$

where

$$b_1 = \frac{a_1 a_4}{a_{13} a_{34}} = \frac{\gamma^2}{(\gamma + \alpha\beta)(1+\gamma)}$$
$$b_2 = \frac{a_2 a_3}{a_{24} a_{34}} = \frac{1}{(1+\gamma)(1+\beta)} \tag{4.22}$$
$$b_3 = \frac{a_3 a_4 a_5}{a_{13} a_{34} a_{24}} = \frac{\beta(\gamma(1+\alpha) + \alpha\beta(1+\gamma))}{(\gamma+\alpha\beta)(1+\gamma)(1+\beta)}.$$

Obviously $b_1 + b_2 + b_3 = 1$ and the normalization property is fulfilled

$$\sum_{km} w_{km} = (b_1 + b_2 + b_3)^N = 1. \tag{4.23}$$

Hence the polynomials $P_{pr}(k, m; N)$ are orthonormal polynomials which are orthogonal with respect to the trinomial distribution. It is natural to call them Krawtchouk polynomials in two discrete variables because the ordinary Krawtchouk polynomials (in one argument) are orthogonal with respect to the binomial distribution [18].

The polynomials $Q_{pr}(k, m; N)$ introduced in (4.1) differ from the orthonormal polynomials $P_{pr}(k, m; N)$ by a simple factor,

$$P_{pr}(k, m; N) = B_{pr} Q_{pr}(k, m; N) \tag{4.24}$$

where the coefficients

$$B_{pr} = (-1)^p \sqrt{\frac{N!}{p!r!(N-p-r)!} \left(\frac{a_1 a_4 a_{34}}{a_2 a_3 a_{12}}\right)^p \left(\frac{a_4 a_5}{a_3 a_{12}}\right)^r}$$
$$= (-1)^p \sqrt{\frac{N!}{p!r!(N-p-r)!} \left(\frac{\beta(1+\gamma)}{1+\alpha}\right)^p \left(\frac{\gamma(1+\alpha) + \alpha\beta(1+\gamma)}{1+\alpha}\right)^r} \tag{4.25}$$

do not depend on the arguments k, m. Hence the polynomials $Q_{pr}(k, m; N)$ are orthogonal with respect to the same trinomial distribution (4.21) but they are not orthonormal. Instead we have the following orthogonality relation for them,

$$\sum_{km} w_{km} Q_{pr}(k, m; N) Q_{p'r'}(k, m; N) = \frac{\delta_{pp'}\delta_{rr'}}{B_{pr}^2} \tag{4.26}$$

where the weight function w_{km} is given by (4.21). Note that the polynomials $Q_{pr}(k, m; N)$ depend on four independent parameters α, β, γ and N and are normalized by the condition $Q_{00}(k, m) = 1$.

5. Factorization chain for the polynomials $Q_{pr}(k, m; N)$

In this section we demonstrate that the well known factorization method [19, 20] (known also as Darboux transformation chain) is a powerful tool for investigation of the polynomials $Q_{pr}(k, m; N)$. In particular, we show that the polynomials $Q_{pr}(k, m; N)$ are eigenfunctions of two commuting difference operators and, moreover, we find the Rodriguez formula for these polynomials. For applications of the factorization method to the theory of orthogonal polynomials in one variable see, for example, [21–24].

The operators $L^{\pm}(N)$ and $M^{\pm}(N)$ introduced in the previous section satisfy the following relations,

$$L^- L^+(N) = L^+(N-1) L^- + \varepsilon \tag{5.1}$$
$$M^- M^+(N) = M^+(N-1) M^- + \varepsilon \tag{5.2}$$

which indicate that these operators obey factorization chain conditions. Indeed (see, e.g., [19, 20]), the abstract algebraic Darboux (factorization) chain for the operator family $J^{\pm}(N)$ depending on the integer parameter N can be written in the form

$$J^{-}(N)J^{+}(N)-\nu(N)=J^{+}(N-1)J^{-}(N-1)-\nu(N-1) \tag{5.3}$$

where $\nu(N)$ are some parameters.

It is seen from (5.1) and (5.2) that in our case we indeed have two factorization chains for the operators $L^{\pm}(N)$ and $M^{\pm}(N)$ with the same parameters

$$\nu(N)=\varepsilon N \tag{5.4}$$

(note that the operators L^{-} and M^{-} do not depend on N).

Given the factorization chain (5.3), one can construct the family of Hamiltonians

$$H(N)=J^{+}(N)J^{-}(N)-\nu(N) \tag{5.5}$$

such that the operators $J^{\pm}(N)$ play the role of Darboux transformations for these Hamiltonians. Let ψ be an eigenstate of the operator $H(N)$ with the eigenvalue λ: $H(N)\psi=\lambda\psi$. Then the state $J^{-}(N)\psi$ is the eigenstate of the Hamiltonian $H(N-1)$ with *the same* eigenvalue: $H(N-1)J^{-}(N)\psi=\lambda J^{-}(N)\psi$. Analogously, the operator $J^{+}(N)$ transforms an eigenstate of the operator $H(N-1)$ into the eigenstate of the operator $H(N)$. This ladder property of the factorization chain allows one to construct a whole family of eigenstates of the Hamiltonian $H(N)$ starting from a given state which can be easily found (for details see [19]).

In our case we have two families of Hamiltonians

$$H_1(N)=L^{+}(N)L^{-}-\varepsilon N \qquad \text{and} \qquad H_2(N)=M^{+}(N)M^{-}-\varepsilon N. \tag{5.6}$$

It is easily verified that the operators $H_1(N)$ and $H_2(N)$ commute with one another (for the same value of N). The eigenvalue problems for these Hamiltonians lead to the *two independent difference equations* for the polynomials $Q_{pr}(k,m;N)$,

$$L^{+}(N)L^{-}Q_{pr}(k,m;N)=\varepsilon p Q_{pr}(k,m;N) \tag{5.7}$$

$$M^{+}(N)M^{-}Q_{pr}(k,m;N)=\varepsilon r Q_{pr}(k,m;N). \tag{5.8}$$

Explicitly, these difference operators can be written as

$$\begin{aligned} L^{+}(N)L^{-} = {} & \frac{\alpha\sigma_1}{\gamma}kT_k^{-}-\frac{m\sigma_1}{\beta}T_m^{-}-\frac{\gamma(\beta-\delta)m}{\beta}T_k^{+}T_m^{-}-\frac{\alpha\beta(1+\gamma)k}{\gamma}T_k^{-}T_m^{+} \\ & -\gamma(\delta-\beta)(N-k-m)T_k^{+}-\beta(1+\gamma)(N-k-m)T_m^{+}+N\sigma_1 \\ & +k(\alpha\beta-\alpha\delta-\sigma_1)+m(\gamma+1-\sigma_1) \end{aligned} \tag{5.9}$$

where $\sigma_1=\beta+\gamma\delta$, and

$$\begin{aligned} M^{+}(N)M^{-} = {} & -\frac{\sigma_2}{\gamma}kT_k^{-}+\frac{m\sigma_2\delta}{\beta}T_m^{-}-\frac{\gamma\delta(\beta+1)m}{\beta}T_k^{+}T_m^{-}-\frac{\beta(\gamma-\alpha)k}{\gamma}T_k^{-}T_m^{+} \\ & -\gamma(\beta+1)(N-k-m)T_k^{+}+\beta(\gamma-\alpha)(N-k-m)T_m^{+}+N\sigma_2 \\ & +k(\beta+1-\sigma_2)+m(\delta(\gamma-\alpha)-\sigma_2) \end{aligned} \tag{5.10}$$

where $\sigma_2=\gamma+\alpha\beta$.

Thus, the polynomials $Q_{pr}(k,m;N)$ are simultaneous eigenfunctions of the two commuting difference operators (5.9) and (5.10).

Note that the relations (4.4), (4.12), (4.7) and (4.13) are nothing other than the explicit form of the Darboux transformations for our polynomials $Q_{pr}(k,m;N)$ because these polynomials are eigenfunctions of the Hamiltonians $H_{1,2}(N)$. Repeating the action of these transformations we obtain

$$L^+(N+j_1)L^+(N+j_1-1)\dots L^+(N+1)Q_{pr}(k,m;N) = (N+1)_{j_1}Q_{p+j_1,r}(k,m;N+j_1) \qquad (5.11)$$

$$M^+(N+j_2)M^+(N+j_2-1)\dots M^+(N+1)Q_{pr}(k,m;N) = (N+1)_{j_2}Q_{p,r+j_2}(k,m;N+j_2) \qquad (5.12)$$

where $(a)_j = a(a+1)\dots(a+j-1)$ denotes the standard Pochhammer symbol. These extended relations allow one to construct a simple Rodriguez-type formula for the polynomials $Q_{pr}(k,m;M)$. Indeed, obviously $Q_{00}(k,m;N) = 1$ (for any values of k,m,N). Hence from (5.11) and (5.12) we have the following formula:

$$M^+(N+j_1+j_2)M^+(N+j_1+j_2-1)\dots M^+(N+j_1+1)L^+(N+j_1)\dots L^+(N+1)(1) = (N+1)_{j_1}(N+j_1+1)_{j_2}Q_{j_1,j_2}(k,m;N+j_1+j_2). \qquad (5.13)$$

The formula (5.13) is nothing other than the Rodriguez-type formula: it allows one to construct any polynomial $Q_{pr}(k,m;N)$ (because the parameters j_1, j_2, N are arbitrary positive integers).

Some remarks should be made concerning the factorization scheme (5.1) and (5.2). In general, the operators $L^{\pm}(N)$ and $M^{\pm}(N)$ belonging to two families do not commute with one another. This leads, for example, to many other versions of the Rodriguez formula (5.13) (these versions are obtained by changing the ordering of the operators L^+ and M^+ in the string on the left-hand side of (5.13)). Nevertheless, the operators $H_1(N)$ and $H_2(N)$ (Hamiltonians) commute with one another, as well as the operators L^- and M^-. Moreover, as is easily seen, the operators L^- and M^- play the role of 'difference derivations': they decrease the degree of *any polynomial* in two variable by one. In this respect, the polynomials $Q_{pr}(k,m;N)$ have almost the same properties as the classical polynomials in one discrete argument (see, e.g., [4]): they satisfy difference equations, difference–difference relations, have a simple Rodriguez-type formula, admit the Darboux transformation chain and are covariant with respect to difference derivation.

We were not able to find a simple explicit expression for the polynomials $Q_{pr}(k,m;N)$; however, in section 8 we show that for a special (symmetric) choice of the representation parameters a_i there is a nice expression for these polynomials in terms of the ordinary Krawtchouk polynomials in one variable.

6. Hidden symmetry algebra of the eigenvalue problem for the polynomials $Q_{pr}(k,m;N)$

In the previous section we introduced the operators $L^{\pm}(N)$ and $M^{\pm}(N)$ and showed that they satisfy factorization chain relations (5.1) and (5.2). Note that these relations connect the operators with *different values* of the parameter N. In this section we consider algebraic properties of the operators $L^{\pm}$ and $M^{\pm}$ belonging to *the same* value of the parameter N. For simplicity, we will omit the explicit dependence of these operators in N in this section.

Using explicit expressions (4.5), (4.14), (4.8) and 4.15) for these operators it is easily verified that they form a linear (Lie) algebra under the commutations:

$$[L^-, M^-] = 0 \qquad [L^-, M^+] = -L^- \qquad [L^+, M^-] = M^-$$
$$[L^+, M^+] = L^+ - M^+ \qquad [L^-, L^+] = \varepsilon - L^- \qquad [M^-, M^+] = \varepsilon - M^-. \qquad (6.1)$$

Introduce the operators $K_1 = L^+L^-$ and $K_2 = M^+M^-$ which play the role of Hamiltonians with eigenstates being just the polynomials $Q_{pr}(k,m;N)$ (see (5.9) and (5.10)). Then it is verified from (6.1) that commutators of the Hamiltonians with operators $L^{\pm}$ and $M^{\pm}$ also form a closed but *nonlinear* algebra:

$$
\begin{aligned}
&[K_1, L^-] = (L^- - \varepsilon)L^- \qquad [K_1, L^+] = \varepsilon L^+ - K_1 \qquad [K_1, M^-] = M^- L^- \\
&[K_1, M^+] = -M^+ L^- \qquad [K_2, L^-] = L^- M^- \qquad [K_2, L^+] = -L^+ M^- \\
&[K_2, M^-] = (M^- - \varepsilon)M^- \qquad [K_2, M^+] = \varepsilon M^+ - K_2.
\end{aligned} \tag{6.2}
$$

First, note that from (6.2) we can again verify that the operators K_1 and K_2 do commute with one another:

$$[K_1, K_2] = [K_1, M^+M^-] = M^+M^-L^- - M^+L^-M^- = 0. \tag{6.3}$$

Moreover, introducing the operators

$$\tilde{L}^+ = L^+ - K_1/\varepsilon \qquad \tilde{M}^+ = M^+ - K_2/\varepsilon \tag{6.4}$$

we get the commutation relations

$$[K_1, \tilde{L}^+] = \varepsilon \tilde{L}^+ \qquad [K_2, \tilde{M}^+] = \varepsilon \tilde{M}^+ \tag{6.5}$$

from which it is seen that the operators $\tilde{L}^+$ and $\tilde{M}^+$ play the role of raising operators with respect to the Hamiltonians K_1 and K_2. However, these operators do not allow one to construct the eigenstates of the Hamiltonians *simultaneously*. This means that if, say ψ, is an eigenstate for the operators K_1 and K_2, then the operator $\tilde{L}^+$ transforms ψ into another eigenstate of the operator K_1 *but not of the operator* K_2. Hence, the operators $\tilde{L}^+$ and $\tilde{M}^+$ are not appropriate for constructing simultaneous eigenstates. Nevertheless, we can introduce the operators $A = L^+M^-$ and $B = M^+L^-$ which are 'true' raising–lowering operators for the simultaneous eigenstates. Indeed, one has commutation relations

$$[K_1, A] = -[K_2, A] = \varepsilon A \qquad [K_1, B] = -[K_2, B] = -\varepsilon B. \tag{6.6}$$

Hence, the operator A is a raising operator for K_1 and a lowering operator for K_2, and the operator B is a raising operator for K_2 and a lowering operator for K_1. Thus, if we have an eigenstate ψ_{pr} of the operators K_1 and K_2 (with eigenvalues εp and εr, respectively) we can construct other eigenstates of these operators by applying the operators A and B:

$$A^{j_1} B^{j_2} \psi_{pr} \propto \psi_{p+j_1-j_2, r+j_2-j_1}. \tag{6.7}$$

Note that obviously $p + r =$ constant under the action of the operators A and B, so we cannot obtain all possible eigenstates by this method starting from a given eigenstate.

The hidden symmetry nonlinear algebra (6.2) underlying the 9j-problem for the oscillator algebra somewhat resembles other hidden symmetry algebras underlying completely integrable many-particle systems (see, e.g., [9]).

7. Duality property

In this section we show that the polynomials $Q_{pr}(k, m; N)$ possess a duality property with respect to exchanging $\{p, r\}$ and $\{k, m\}$. This can be derived, for example, by means of dual difference–difference relations for these polynomials. Denote

$$X = 1 - v - \alpha u \qquad Y = 1 + \beta v - \gamma u \qquad Z = 1 + u + \delta v.$$

Then from the identities

$$(\delta - \beta)X - (1 + \delta)Y + (1 + \beta)Z = \varepsilon u \tag{7.1}$$

$$-(1 + \gamma)X + (1 + \alpha)Y + (\gamma - \alpha)Z = \varepsilon v \tag{7.2}$$

we get the following relations for the polynomials $Q_{pr}(k, m; N)$

$$\gamma(\beta-\delta)Q_{p+1,r}(k,m;N)-\gamma(1+\beta)Q_{p,r+1}(k,m;N)+\gamma(1+\delta)Q_{pr}(k,m;N) = (k\varepsilon/N)Q_{pr}(k-1,m;N-1) \tag{7.3}$$

$$-\beta(\gamma+1)Q_{p+1,r}(k,m;N)+\beta(1+\alpha)Q_{p,r}(k,m;N)+\beta(\gamma-\alpha)Q_{p,r+1}(k,m;N) = (m\varepsilon/N)Q_{pr}(k,m-1;N-1). \tag{7.4}$$

Analogously from the identities

$$\Psi_u^{pr}(u,v;N) = \left(-\frac{\alpha p}{X} - \frac{\gamma q}{Y} + \frac{r}{Z}\right)\Psi^{pr}(u,v;N) \tag{7.5}$$

$$\Psi_v^{pr}(u,v;N) = \left(-\frac{p}{X} + \frac{\beta q}{Y} + \frac{\delta r}{Z}\right)\Psi^{pr}(u,v;N) \tag{7.6}$$

we get another pair of dual relations

$$(\alpha/\gamma)pQ_{p-1,r}(k,m;N-1)-(r/\gamma)Q_{p,r-1}(k,m;N-1)+(N-p-r)Q_{pr}(k,m;N-1) = NQ_{pr}(k+1,m;N) \tag{7.7}$$

and

$$(-p/\beta)Q_{p-1,r}(k,m;N-1)+(\delta r/\beta)Q_{p,r-1}(k,m;N-1)+(N-p-r)Q_{pr}(k,m;N-1) = NQ_{pr}(k,m+1;N). \tag{7.8}$$

Relations (7.3) and (7.4) are dual analogies of relations (4.12) and (4.13), whereas relations (7.7) and (7.8) are dual analogies of relations (4.4) and (4.7). It is easily verified that dual relations are obtained from initial ones by changing of the variables and parameters

$$k \to p \qquad m \to r \qquad \beta \to \gamma \qquad \gamma \to \beta \qquad \alpha \to \frac{\alpha\beta}{\gamma}. \tag{7.9}$$

Taking into account the fact that

$$Q_{00}(k,m;N) = Q_{pr}(0,0;N) = 1 \tag{7.10}$$

we arrive at the important duality property for the polynomials themselves

$$Q_{pr}(k,m;\alpha,\beta,\gamma;N) = Q_{km}\left(p,r;\frac{\alpha\beta}{\gamma},\gamma,\beta;N\right) \tag{7.11}$$

where in (7.11) we have inserted the dependence of the polynomials on the parameters α, β, γ.

The duality property (7.11) can be explained from the following considerations. The $9j$-symbols satisfy the relation (2.17). From unitarity of this transformation and reality of $9j$-symbols we obtain the dual decomposition

$$|*;k,l,m\rangle = \sum_{pqr} F_{pqr}^{klm}|*;p,q,r\rangle. \tag{7.12}$$

On the other hand, decomposition (7.12) can be obtained from (2.17) by formally exchanging $A^{(2)} \leftrightarrow A^{(3)}$ of the oscillator algebras whereas the algebras $A^{(1)}$ and $A^{(4)}$ remain unchanged. This algebras' exchange is equivalent just to exchange the parameters (7.9).

It is seen from (7.11) that if $\beta = \gamma$ then the polynomials $Q_{pr}(k, m; N)$ are *self-dual*

$$Q_{pr}(k,m;\alpha,\beta,\beta;N) = Q_{km}(p,r;\alpha,\beta,\beta;N). \tag{7.13}$$

This self-duality property is also obvious from the addition scheme, because the condition $\beta = \gamma$ means that $a_2 = a_3$; hence, interchanging algebras $A^{(2)}$ and $A^{(3)}$ have the same representation parameters.

Note that from duality relation (7.11) for the polynomial $Q_{pr}(k, m; N)$ we can obtain two independent recurrence relations for these polynomials which are dual analogies of the difference equations (5.7) and (5.8). We will not write down these relations because they are obvious.

8. Explicit expression for the symmetric case

In this section we consider a special 'symmetric' case when the algebras' constants satisfy the restriction

$$a_1 a_4 = a_2 a_3. \tag{8.1}$$

Then $\delta = 0$, $\alpha = \gamma$ and the generating function becomes

$$\Psi(u, v) = (1+u)^r (1 - v - \alpha u)^p (1 + \beta v - \alpha u)^q. \tag{8.2}$$

We can expand this function using the well known generating relation for the ordinary Krawtchouk polynomials [18]

$$(1+z)^y (1 - \tau z)^x = \sum_{n=0}^{x+y} \binom{x+y}{n} K_n(x; \tau; x+y) z^n \tag{8.3}$$

where x and y are some positive integers, τ is a positive parameter and the Krawtchouk polynomials in argument x have the expression

$$K_n(x; \tau; M) = {}_2F_1\left(\begin{matrix} -x, -n \\ -M \end{matrix}; \tau + 1\right). \tag{8.4}$$

Note an obvious duality property

$$K_n(x; \tau; M) = K_x(n; \tau; M). \tag{8.5}$$

Now we can rewrite the generating function $\Psi(u, v)$ in the form

$$\Psi(u, v) = (1+u)^r (1 - \alpha u)^{p+q} (1 - z/\beta)^p (1+z)^q \tag{8.6}$$

where

$$z = \frac{a_3 v}{a_1 - a_2 u}. \tag{8.7}$$

Then we have successively

$$\begin{aligned}
\Psi(u, v) &= (1+u)^r (1 - \alpha u)^{p+q} \sum_{m=0}^{p+q} \binom{p+q}{m} (1 - \alpha u)^{-m} K_m(p; a_1/a_3; p+q)(\beta v)^m \\
&= \sum_{m=0}^{p+q} \sum_{k=0}^{N-m} \binom{p+q}{m} \binom{N-m}{k} K_m(p; a_1/a_3; p+q) \\
&\quad \times K_k(p+q-m; a_2/a_1; N-m) u^k (\beta v)^m.
\end{aligned} \tag{8.8}$$

Thus from (8.8), (3.10) and symmetry property (8.5) of the Krawtchouk polynomials we get the explicit expression for modified $9j$-symbols:

$$\Phi_{klm}^{pqr} = (a_3/a_1)^m \binom{p+q}{m} \binom{k+l}{k} K_p(m; a_1/a_3; p+q) K_{p+q-m}(k; a_2/a_1; N-m). \tag{8.9}$$

From (4.1) and (4.2) we obtain the expression for the corresponding polynomials $Q_{pr}(k, m; N)$:

$$Q_{pr}(k, m; N) = \frac{(N-r)!(N-m)!}{N!(N-r-m)!} {}_2F_1\left(\begin{matrix} -m, -p \\ r-n \end{matrix}; a_{13}/a_3\right) {}_2F_1\left(\begin{matrix} -k, -r \\ m-N \end{matrix}; a_{12}/a_2\right). \tag{8.10}$$

It is seen from (8.10) (or (8.9)) that $Q_{pr}(k, m; N)$ are indeed polynomials in two discrete arguments k, m. These polynomials are orthogonal with respect to the weight function (4.21) where in the symmetric case

$$\begin{aligned} b_1 &= \frac{a_1 a_2}{a_{12} a_{13}} = \frac{\alpha}{(1+\beta)(1+\alpha)} \\ b_2 &= \frac{a_1^2}{a_{12} a_{13}} = \frac{1}{(1+\alpha)(1+\beta)} \\ b_3 &= \frac{a_3}{a_{13}} = \frac{\beta}{1+\beta}. \end{aligned} \tag{8.11}$$

It is interesting to note that the Krawtchouk polynomials (8.10) in two discrete variables were studied by Prizva [13], who also found a generating function similar to (8.2). These polynomials can also be obtained by a limiting process from the Hahn polynomials in two variables found by Dunkl [25].

Perhaps the polynomials (8.10) provide the first *explicit* example of two-variable orthogonal polynomials connected with $9j$-symbols. Surprisingly, a simple explicit expression arises only for the symmetric case (8.1). The reason for this phenomenon is yet unclear.

9. 9j-symbols and Appell hypergeometric function

In this section we return to the general case of the representation parameters a_i (i.e. $\delta \neq 0$ in (3.10)). When one of the parameters p, q, r is equal to zero then it is possible to express the corresponding $9j$-symbols in terms of the Appell hypergeometric function in two variables.

Indeed, consider, for example, the case $q = 0$. Then obviously $p + r = N$. Assume additionally that

$$k + m \leqslant r. \tag{9.1}$$

Then we have the expansion

$$\begin{aligned} \Psi(u, v) &= (1 - v - \alpha u)^p (1 + u + \delta v)^r \\ &= \sum_{n_1, n_2, j_1, j_2} \frac{(-\alpha u)^{n_1} (-v)^{n_2} u^{j_1} (\delta v)^{j_2} p! r!}{n_1! n_2! (p - n_1 - n_2)! j_1! j_2! (r - j_1 - j_2)!} \end{aligned} \tag{9.2}$$

where summation is restricted by the conditions $n_1 + n_2 \leqslant p$, $j_1 + j_2 \leqslant r$. Introducing discrete variables $k = n_1 + j_1$ and $m = n_2 + j_2$ we can rewrite the sum (9.2) in the form

$$\Psi(u, v) = \sum_{n_1, n_2, k, m} \frac{p! r! (-\alpha)^{n_1} (-1/\delta)^{n_2} u^k (\delta v)^m}{n_1! n_2! (k - n_1)! (m - n_2)! (p - n_1 - n_2)! (r - k - m + n_1 + n_2)!}. \tag{9.3}$$

Under the restriction (9.1) we can transform (9.3) to

$$\Psi(u, v) = \sum_{k,m} \frac{r! u^k (\delta v)^m}{k! m! (r - k - m)!} \sum_{n_1, n_2} \frac{(-k)_{n_1} (-m)_{n_2} (-p)_{n_1+n_2} (-\alpha)^{n_1} (-1/\delta)^{n_2}}{n_1! n_2! (1 + r - k - m)_{n_1+n_2}}. \tag{9.4}$$

The inner sum in (9.4) can be expressed as

$$\begin{aligned} &\sum_{n_1, n_2} \frac{(-k)_{n_1} (-m)_{n_2} (-p)_{n_1+n_2} (-\alpha)^{n_1} (-1/\delta)^{n_2}}{n_1! n_2! (1 + r - k - m)_{n_1+n_2}} \\ &\qquad = F_1(-p, -k, -m, 1 + r - k - m; -\alpha, -1/\delta) \end{aligned} \tag{9.5}$$

where $F_1(a, b, c, d; x, y)$ is the Appell hypergeometric function in two variables x, y defined by [26, vol 1]

$$F_1(a, b, c, d; x, y) = \sum_{n_1, n_2} \frac{(a)_{n_1+n_2}(b)_{n_1}(c)_{n_2} x^{n_1} y^{n_2}}{n_1! n_2! (d)_{n_1+n_2}} \tag{9.6}$$

(generally the double sum in (9.6) is infinite; however if, as in our case, $a = -M$ where M is a positive integer then the sum is terminated).

Thus we have the explicit expression for the modified 9 j-symbols in this special case:

$$\Phi_{klm}^{p0r} = \frac{r! \delta^m}{k! m! (r-k-m)!} F_1(-p, -k, -m, 1+r-k-m; -\alpha, -1/\delta). \tag{9.7}$$

The Appell function F_1 has many nice properties (see, for example, the integral representation in [26, vol 1]); however, in general it does not admit the expression in terms of 'more elementary' functions of one variable. It would be interesting to find an expression of 9 j-symbols (in the general case of arbitrary p, q, r) in terms of more complicated multivariable hypergeometric functions (say, Lauricella functions etc).

Note that the expression (9.7) is valid provided the restriction (9.1) is fulfilled. For other regions of the parameters k and m one can also reduce the summation to the Appell hypergeometric function; however, we will not write down the corresponding explicit formulae here because they are obtained by almost the same procedure.

10. Conclusion

We have obtained a simple generating function for 9 j-symbols of the oscillator algebra. This allows one to reconstruct many important properties of the polynomials $Q_{pr}(k, m; N)$ in two discrete variables: the weight function, the difference equation, the difference–difference recurrence relations etc. We would like to mention an interesting property of the polynomials $Q_{pr}(k, m; N)$: they are eigenfunctions of two commuting difference operators K_1 and K_2 (f-las (5.7) and (5.8)). This resembles the situation with completely integrable many-particle systems where the corresponding eigenfunctions can be expressed in terms of some orthogonal polynomials in several variables. These polynomials are eigenfunctions of a set of commuting differential or difference operators, one of them being the Hamiltonian of the system (see, e.g., [7]).

In general, we did not find a simple explicit representation for the polynomials $Q_{pr}(k, m; N)$. Nevertheless, in the symmetric case $a_1 a_4 = a_2 a_3$ a surprising simplification emerges and the polynomials $Q_{pr}(k, m; N)$ can be written as a product of the two ordinary Krawtchouk polynomials (formula (8.9)). The nature for such a simplification is puzzling, because *a priori* there are no arguments for why this symmetry condition is more preferable among other possible restrictions.

There are other interesting questions connected with the general theory of orthogonal polynomials (OPs) in two arguments. For example, it is well known that for the case of one argument the weight function uniquely defines a system of a corresponding OP (up to normalization constants). This is not the case for the OP in two (and more) arguments. Indeed, the generic trinomial distribution (4.21) leads to quite different systems of OPs in symmetric and non-symmetric cases (it is clear that the parameters b_1, b_2, b_3 remain arbitrary in the symmetric case (8.11) excepting the restriction $b_1+b_2+b_3 = 1$). This phenomenon is well known in the general theory of OPs in two variables (see, e.g., [26, vol 2]). However, it would be interesting to investigate under which relations the system of OPs becomes uniquely defined.

Existence of the factorization chain for the polynomials $Q_{pr}(k, m; N)$ described by the operator relations (5.1) and (5.2) is another nice property of the corresponding polynomials allowing an explicit construction of the Rodriguez-type formula. Note that recently some multivariate OPs (appearing in the theory of integrable systems) were shown to possess a Rodriguez formula (see, e.g., [8]); however, as far as we know, the factorization method was not exploited with respect to multivariate polynomials.

Existence of the duality property (7.11) is also an important property, which has an exact analogue for the ordinary Krawtchouk polynomials.

Another interesting aspect is the appearance of the Appell hypergeometric function F_1 in the expression of $9j$-symbols for the special choice $q = 0$. Note that in [27] $9j$-symbols of the $SU(2)$ group were expressed in terms of some formal triple hypergeometric series. One should expect that generic $9j$-symbols for the oscillator algebra are also expressible in terms of some triple hypergeometric series, because the oscillator algebra can be obtained by a simple contraction from the $su(2)$ algebra [16]. We mention the interesting works [28, 29] where some special classes of the Krawtchouk (and some other) multivariable polynomials were expressed in terms of Lauricella polynomials F_B.

Note finally that in [30] and [31] some multivariate Racah polynomials were introduced associated with the multiplicity free Racah coefficients for the Lie group $U(n)$. It would be interesting to recognize possible connections with these objects. We believe, however, that Krawtchouk polynomials connected with oscillator $9j$-symbols are not related with polynomials introduced in [30] and [31], because the latter are invariant under permutation of variables whereas our Krawtchouk polynomials do not possess such a property.

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